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## Localization in one-dimensional random random walks

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Received 16 January 1998

**Abstract.** Diffusion in a one-dimensional random force field leads to interesting localization effects, which we study using the equivalence with a directed walk model with traps. We show that although the average dispersion of positions  $\langle x^2 \rangle - \langle x \rangle^2$  diverges for long times, the probability that two independent particles occupy the same site tends to a finite constant in the small bias phase of the model. Interestingly, the long-time properties of this off-equilibrium, ageing phase is similar to the equilibrium phase of the random energy model.

### 1. Introduction

The properties of random walks in random environments can be markedly different from those of homogeneous random walks [1]. For example, the typical distance travelled by a diffusing particle in an unbiased random force field in one dimension grows with time as  $x \propto \log^2 t$ , instead of the usual  $\sqrt{t}$  law [2]. This is due to the fact that the potential energy typically grows as  $\sqrt{x}$ , leading to very high barriers which slow down the progression of the particle. More strikingly, Golosov has shown that the relative distance between two independent particles in the same random force field remains *finite* even for large times [3], whereas it also grows as  $\sqrt{t}$  in a homogeneous medium. This remarkable classical localization phenomenon is due to the fact that the 'best' potential minimum which can be reached by the particles after a long time  $t$  is so much better than the 'second best' that all the particles have time to gather there, before eventually moving to an even better location.

In the presence of a non-zero average bias  $F_0 > 0$ , several regimes must still be distinguished. For small enough  $F_0$ , the mean position of the particles grows as  $t^\mu$ , where the exponent  $\mu < 1$  is proportional to  $F_0$  [1, 4–6]. Beyond a critical force, the particles move to the right with a non-zero velocity. However, the dispersion around the mean velocity is still anomalous until  $\mu$  reaches the value  $\mu = 2$ . Beyond  $\mu = 2$ , the spreading is 'normal', i.e. Gaussian with a width growing as  $\sqrt{t}$ . The question we wish to address in this paper is whether the Golosov phenomenon survives in the presence of a non-zero average force. We shall actually show that different 'localization' criteria lead to different answers: while the average width of a packet of independent particles diverges with time, there is a finite probability (even at long times) that two particles are nearby in space. The physical picture is that the density of particles is concentrated on a finite number of sites, but the relative distance between these peaks grows with time. As we shall also discuss, there is a strong analogy between this problem and the low-temperature phase of Derrida's random energy model [7].

## 2. Model and simulation

Actually, one can map the long-time behaviour of the problem onto that of a much simpler directed walk model [1, 6, 8, 9], where each particle hops to the right on an even-spaced discrete lattice, with a site-dependent hopping rate  $W_n$  distributed as

$$\rho(W) = \frac{1}{\Gamma(\mu)} W^{\mu-1} e^{-W}. \quad (1)$$

For  $\mu < 1$  the average trapping time  $\overline{1/W}$  is infinite, this leads to the anomalous behaviour of the particles average position reported above. The probability to find the particle on site  $n$  obeys the following master equation:

$$\frac{dP_n(t)}{dt} = -W_n P_n(t) + W_{n-1} P_{n-1}(t). \quad (2)$$

The properties of equation (2) were discussed in detail in [9, 10]. In particular, one can compute the average position of the particle, defined as

$$\overline{\langle x(t) \rangle} \equiv \sum_{n=0}^{\infty} n \overline{P_n(t)} \quad (3)$$

where the overbar denotes the average over the  $W$ 's. For  $\mu < 1$ , one easily finds  $\overline{\langle x(t) \rangle} = \sin(\pi\mu)/[\pi\mu\Gamma(\mu+1)]t^\mu$  at large times [1]. One can also compute the average width  $\Delta^2$  of the packet, defined as

$$\Delta^2 \equiv \overline{\langle x^2 \rangle} - \langle \overline{x} \rangle^2 \equiv \sum_{n=0}^{\infty} n^2 \overline{P_n(t)} - \left( \sum_{n=0}^{\infty} n \overline{P_n(t)} \right)^2. \quad (4)$$

As shown in [9], the width grows to infinity as

$$\Delta^2 = C(\mu)t^{2\mu} \quad (5)$$

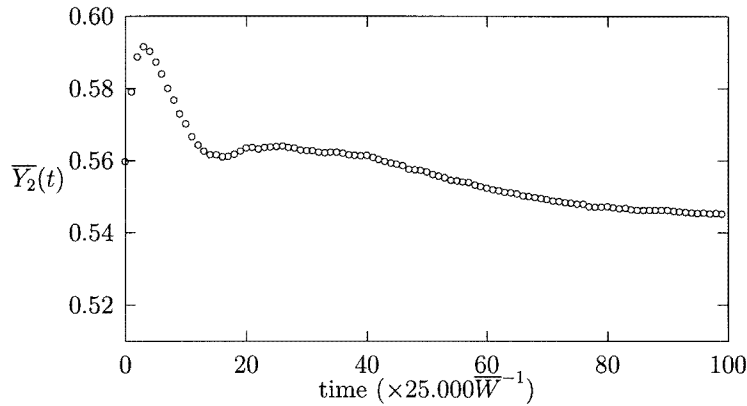
where  $C(\mu)$  is a certain  $\mu$ -dependent number which can be explicitly computed [9], and which goes to zero for  $\mu = 0$ . Naively, this means that the particles' relative positions become further apart (for  $\mu > 0$ ) as  $t$  becomes large, at variance with Golosov's result for the unbiased case, which shows that  $\langle x^2 \rangle - \langle \overline{x} \rangle^2$  remains finite for large  $t$ . However, one can still ask the following question: what is the total probability that two particles, initially at site  $n = 0$ , occupy the same site after time  $t$ ? This is obtained as

$$Y_2(t) = \sum_{n=0}^{\infty} P_n^2(t). \quad (6)$$

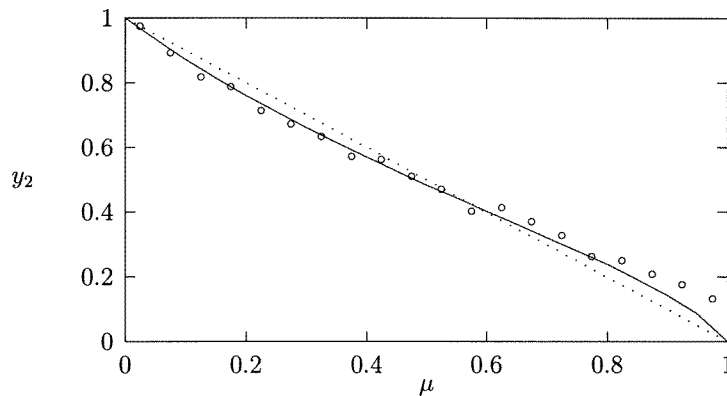
The notation  $Y_2$  is introduced in analogy with spin glasses, where the same question is asked about two copies (replicas) of the same system in equilibrium, and it measures the probability that these two copies occupy the same state [11]. Note that  $Y_2$  is also often taken to be an indicator of localization in quantum problems, where  $P_n = |\psi_n|^2$  is the quantum probability of presence [12]. One can actually study generalized objects, such as

$$Y_q(t) = \sum_{n=0}^{\infty} P_n^q(t) \quad (7)$$

which measure the probability that  $q$  particles occupy the same site. We have numerically studied  $Y_2$  and  $Y_3$  using Monte Carlo methods for the problem defined by equations (1) and (2). In figure 1, we show the average value  $\overline{Y_2(t)}$  as a function of  $t$  for  $\mu = 0.4 < 1$ . This quantity clearly tends to a non-zero constant  $y_2(\mu)$ , which we plot as a function of  $\mu$  in figure 2, together with our theoretical prediction (see below).



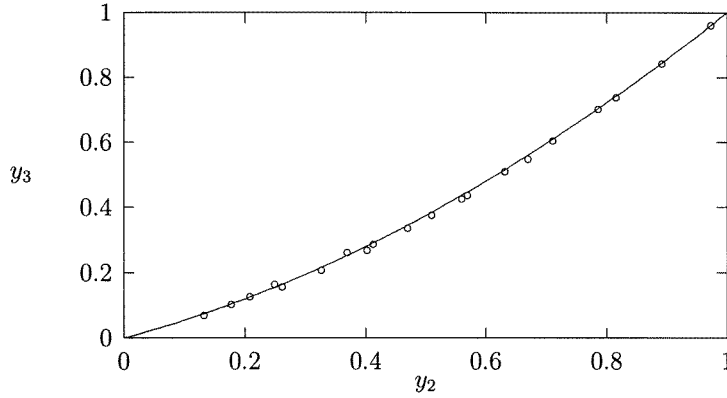
**Figure 1.** Time evolution of the quantity  $\bar{Y}_2(t)$  in the Monte Carlo simulation of a directed random model with hopping rates distributed according to a gamma distribution of index  $\mu = 0.4$ . The simulation was carried out in a 20 000 point lattice with 1000 particles and averaged over 250 disorder samples. Until the last observation times more than 99% of the particles remained within the lattice. This curve shows that  $\bar{Y}_2(t)$  tends to a non-zero constant for asymptotic times.



**Figure 2.**  $y_2$  versus  $\mu$  as given by numerical simulations (small circles), the analytic formula obtained in the text (full curve) and by the equilibrium result of the associated random energy model,  $y_2 = 1 - \mu$ , (dotted curve). The agreement with the theoretical prediction is good, except for values near  $\mu = 1$ : the observation time being finite, we expect to overestimate  $y_2(\mu)$  increasingly more the closer one gets to  $\mu = 1$  since the approach to the actual asymptotic value becomes very slow (logarithmic) for  $\mu = 1$ .

We see that  $y_2(\mu \rightarrow 0) = 1$ , as expected from Golosov's results, while we observe the tendency  $y_2(\mu \rightarrow 1) \rightarrow 0$ , although our numerical data is biased in this limit. In figure 3, we show the parametric plot of  $y_3(\mu)$  (i.e. the asymptotic value of  $\bar{Y}_3(t)$ ) versus  $y_2(\mu)$ . Interestingly, the resulting curve is seen to be very close to  $y_3 = y_2(1 + y_2)/2$  obtained within the so-called 'one-step replica symmetry breaking' solution of equilibrium random systems.

Before giving a more detailed physical interpretation of these results, we first turn to an analytic calculation of  $y_2(\mu)$ .



**Figure 3.** Representation of  $y_3$  versus  $y_2$  as obtained from our simulation (small circles) and as given by the relation  $y_3 = y_2(1 + y_2)/2$ , characteristic of replica symmetry breaking (full curve). The agreement is remarkable and hints to some deep relation between the present dynamical problem and the equilibrium phase of disordered systems.

### 3. Analytic derivation of $y_2(\mu)$

This section contains rather technical details which can be skipped in a first reading. The physical discussion can be found in the next section.

In the trapping model that we study, the time that the particle sojourns in the  $i$ th trap is given by  $t_i = u_i \tau_i$  where  $\tau_i$  is the characteristic trapping time of the  $i$ th trap ( $\tau_i = W_i^{-1}$ ) and  $u_i$  is an exponentially distributed variable accounting for the individual thermal behaviour of the particle. It is then straightforward to write the following equation for the probability of a particle being at site  $n$  at time  $t$ , given a realization of the disorder  $\{\tau_i\}$  (or, equivalently,  $\{W_i\}$ ):

$$P_n(t) = \int \prod_i^n du_i \exp\left(-\sum_j^n u_j\right) \theta\left(t - \sum_i^{n-1} u_i \tau_i\right) \theta\left(\sum_i^n u_i \tau_i - t\right) \quad (8)$$

$\theta$  being the Heaviside function. Equation (8) says that  $P_n(t)$  is the sum of the probabilities of all the possible thermal histories such that the particle has already made  $n - 1$  jumps up to time  $t$  but not yet  $n$ . Since the local residence times are exponentially distributed with rates  $W_n$ ,  $P_n$  as given in (8) is a solution of equation (2).

We are interested in the probability that two particles, starting together at  $t = 0$  remain at the same site after a sufficiently long time. Therefore, the quantity of interest is the probability that two particles are at site  $n$  at time  $t$ :

$$P_n^2(t) = \int \prod_i^n du_i dv_i \exp\left(-\sum_j^n (u_j + v_j)\right) \theta\left(t - \sum_i^{n-1} u_i \tau_i\right) \theta\left(\sum_i^n u_i \tau_i - t\right) \\ \times \theta\left(t - \sum_i^{n-1} v_i \tau_i\right) \theta\left(\sum_i^n v_i \tau_i - t\right) \quad (9)$$

or, in the Laplace domain,

$$P_n^2(E) = \int \prod_i^n du_i dv_i \exp\left(-\sum_j^n (u_j + v_j)\right) \int_{\max(\sum_i^{n-1} u_i \tau_i, \sum_i^{n-1} v_i \tau_i)}^{\min(\sum_i^n u_i \tau_i, \sum_i^n v_i \tau_i)} e^{-Et} dt. \quad (10)$$

Integrating and using  $\theta$  functions we can rewrite this last expression as

$$\begin{aligned}
 EP_n^2(E) = 2 \int \prod_i^n du_i dv_i & \left\{ \left[ \exp\left(-E \sum_i^{n-1} u_i \tau_i\right) - \exp\left(-E \sum_i^n u_i \tau_i\right) \right] \right. \\
 & \times \theta\left(\sum_i^{n-1} (u_i - v_i) \tau_i\right) \theta\left(\sum_i^n (v_i - u_i) \tau_i\right) \theta\left(\sum_i^n u_i \tau_i - \sum_i^{n-1} u_i \tau_i\right) \\
 & + \left[ \exp\left(-E \sum_i^{n-1} u_i \tau_i\right) - \exp\left(-E \sum_i^n v_i \tau_i\right) \right] \theta\left(\sum_i^{n-1} (u_i - v_i) \tau_i\right) \\
 & \left. \times \theta\left(\sum_i^n (u_i - v_i) \tau_i\right) \theta\left(\sum_i^n v_i \tau_i - \sum_i^{n-1} u_i \tau_i\right) \right\} \exp\left(-\sum_j^n (u_j + v_j)\right) \quad (11)
 \end{aligned}$$

where the three  $\theta$ 's in each summand implement the maximum condition in the lower limit of the integral, the minimum condition in the upper limit and the condition that the upper limit is greater than the lower limit in (10), respectively. In (11) we have also used the symmetrical integration with respect to  $u_i$  and  $v_i$  to somewhat simplify the expression (whence the factor 2).

It can now be proved that expression (11) is equivalent to a much simpler formula, where only one  $\theta$  function appears per summand and this is accomplished by conveniently renaming  $u_i \leftrightarrow v_i$  as integration variables for some terms and by bearing in mind that  $u_i$ ,  $v_i$ , and  $\tau_i$  are all positive, whence the summations are all monotonous increasing functions of the step number  $n$ . This procedure leads to

$$\begin{aligned}
 EP_n^2(E) = 2 \int \prod_i^n du_i dv_i & \exp\left(-\sum_i^n (u_i + v_i)\right) \\
 & \times \left[ \exp\left(-E \sum_i^{n-1} u_i \tau_i\right) \theta\left(\sum_i^{n-1} (u_i - v_i) \tau_i\right) \right. \\
 & - \exp\left(-E \sum_i^n u_i \tau_i\right) \theta\left(\sum_i^n (v_i - u_i) \tau_i\right) \\
 & - \exp\left(-E \sum_i^{n-1} u_i \tau_i\right) \theta\left(\sum_i^{n-1} u_i \tau_i - \sum_i^n v_i \tau_i\right) \\
 & \left. + \exp\left(-E \sum_i^n u_i \tau_i\right) \theta\left(\sum_i^{n-1} v_i \tau_i - \sum_i^n u_i \tau_i\right) \right]. \quad (12)
 \end{aligned}$$

In order to proceed we now use the following representation of the  $\theta$  function in the complex plane

$$\theta(x) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{\lambda} d\lambda \quad (13)$$

and (12) turns into

$$\begin{aligned}
 EP_n^2(E) = -\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} & \int \prod_i^n du_i dv_i \exp\left(-\sum_i^n (u_i + v_i)\right) \\
 & \times \left[ \exp\left(-\sum_i^{n-1} [Eu_i - i\lambda(u_i - v_i)] \tau_i\right) \right]
 \end{aligned}$$

$$\begin{aligned}
& - \exp \left( - \sum_i^n [Eu_i - i\lambda(v_i - u_i)]\tau_i \right) \\
& - \exp \left( - \sum_i^{n-1} [Eu_i - i\lambda(u_i - v_i)]\tau_i \right) e^{-i\lambda v_n \tau_n} \\
& + \exp \left( - \sum_i^{n-1} [Eu_i - i\lambda(v_i - u_i)]\tau_i \right) e^{-(E-i\lambda)u_n \tau_n} \Big]. \tag{14}
\end{aligned}$$

This formulation of the equation permits the factorization, within each summand, of the contributions of each  $du_i dv_i$  so as to obtain

$$\begin{aligned}
EP_n^2(E) &= -\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} \left[ \prod_i^{n-1} \int du dv e^{-u-v} e^{-[Eu-i\lambda(u-v)]\tau_i} \right. \\
& - \prod_i^n \int du dv e^{-u-v} e^{-[Eu-i\lambda(v-u)]\tau_i} \\
& - \int dv_n e^{-(1+i\lambda\tau_n)v_n} \prod_i^{n-1} \int du dv e^{-u-v} e^{-[Eu-i\lambda(u-v)]\tau_i} \\
& \left. + \int du_n e^{-(1+E\tau_n+i\lambda\tau_n)u_n} \prod_i^{n-1} \int du dv e^{-u-v} e^{-[Eu-i\lambda(v-u)]\tau_i} \right]. \tag{15}
\end{aligned}$$

If we now define the following functions

$$F(E, \lambda) = \int du dv e^{-u-v} \overline{e^{-[Eu-i\lambda(u-v)]\tau_i}} \tag{16}$$

$$G(E, \lambda) = \int du e^{-u} \overline{e^{-(Eu+i\lambda u)\tau_i}} \tag{17}$$

where the bar over the exponential represents the average over the possible values of  $\tau_i$ , the disorder average of (15) is readily written as

$$E\overline{P_n^2}(E) = -\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} \{ [1 - G(0, \lambda)] F(E, \lambda)^n - [F(E, -\lambda) - G(E, \lambda)] F(E, -\lambda)^n \}. \tag{18}$$

We now sum (18) for all the values of  $n$  in order to obtain the function  $\overline{Y_2}(E)$ :

$$\overline{Y_2}(E) = \frac{-i}{\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} \left[ \frac{1 - G(0, \lambda)}{1 - F(E, \lambda)} - \frac{F(E, -\lambda) - G(E, \lambda)}{1 - F(E, -\lambda)} \right]. \tag{19}$$

Since  $\tau_i = W_i^{-1}$  the functions  $F(E, \lambda)$  and  $G(E, \lambda)$  can be expressed in terms of the distribution of hopping rates  $\rho(W)$ :

$$\begin{aligned}
F(E, \lambda) &= \int_0^{\infty} dW \rho(W) \frac{W}{W + E - i\lambda} \frac{W}{W + i\lambda} \\
&= 1 - \frac{\lambda^2}{E - 2i\lambda} \int_0^{\infty} dW \frac{\rho(W)}{W + i\lambda} - \frac{(E - i\lambda)^2}{E - 2i\lambda} \int_0^{\infty} dW \frac{\rho(W)}{W + E - i\lambda} \tag{20}
\end{aligned}$$

$$\begin{aligned}
G(E, \lambda) &= \int_0^{\infty} dW \rho(W) \frac{W}{W + E + i\lambda} \\
&= 1 - (E + i\lambda) \int_0^{\infty} dW \frac{\rho(W)}{W + E + i\lambda}. \tag{21}
\end{aligned}$$

Using these expressions in (19) we obtain, after some algebra,

$$E\overline{Y}_2(E) = \frac{2}{\pi} \int_0^\infty \text{Re} \left[ \frac{f(iEu) - f(E - iEu)}{u^2 f(iEu) + (1 - iu)^2 f(E - iEu)} (1 - iu) \right] du \tag{22}$$

the function  $f(z)$  being given by the integral

$$f(z) = \int_0^\infty dW \rho(W) \frac{1}{W + z}.$$

The result (22) is now straightforwardly applied to the relevant distribution of hopping rates  $\rho(W)$ . For instance, it is reassuring to see that for a non-disordered lattice,  $\rho(W) = \delta(W - W_0)$ , equation (22) yields

$$\overline{Y}_2(t) \simeq \frac{1}{2\sqrt{\pi W_0 t}}$$

which can be obtained directly since the two particles are independent, and this means that the probability that the two particles are on the same site is inversely proportional to the typical distance.

For the case that we are exploring here, we focus on a distribution of the kind (1), whence the function  $f(Ez)$  turns out to be

$$f(Ez) = \int_0^\infty dW \frac{W^{\mu-1} e^{-W}}{\Gamma(\mu)} \frac{1}{W + Ez} \simeq E^{\mu-1} \int_0^\infty dx \frac{x^{\mu-1}}{\Gamma(\mu)} \frac{1}{x + z}$$

assuming in the last equality that  $E$  is sufficiently small so as to neglect the factor  $e^{-Ex}$  within the integral. Some further developments making use of the definition of gamma functions allow one to find

$$f(Ez) \simeq \Gamma(1 - \mu) E^{\mu-1} z^{\mu-1} \quad \text{as } E \rightarrow 0.$$

Introducing this expression in (22) we finally obtain

$$E\overline{Y}_2(E) \simeq \frac{2}{\pi} \int_0^\infty \text{Re} \left[ \frac{(iu)^{\mu-1} - (1 - iu)^{\mu-1}}{(1 - iu)^{\mu+1} - (iu)^{\mu+1}} (1 - iu) \right] du \tag{23}$$

which is indeed a finite integral when  $0 < \mu < 1$ . Equation (23) can now be trivially transformed to the time domain again and we obtain a  $\mu$ -dependent constant asymptotic result:

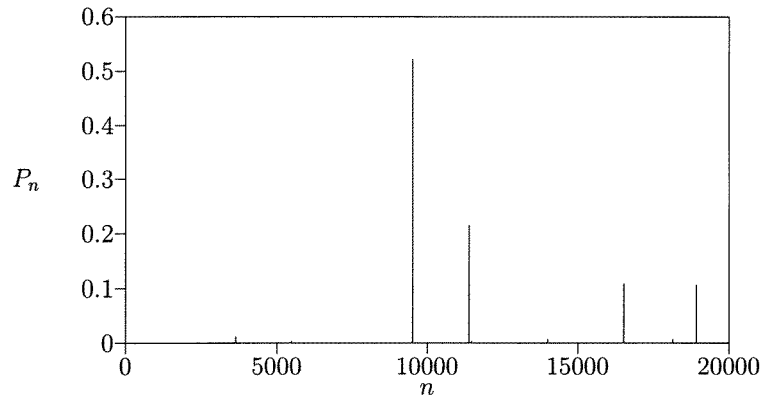
$$y_2(\mu) = \frac{2}{\pi} \int_0^\infty \text{Re} \left[ \frac{(iu)^{\mu-1} - (1 - iu)^{\mu-1}}{(1 - iu)^{\mu+1} - (iu)^{\mu+1}} (1 - iu) \right] du. \tag{24}$$

We have calculated (24) numerically for different values of  $\mu$  in the interval of interest and we have compared the results with the data obtained from the simulation in figure 2. The agreement is quite good except for values near  $\mu = 1$ , where the results of the simulation turn out to be less reliable because of the slow relaxation to the actual asymptotic value in (24).

#### 4. Discussion

How can one reconcile the fact that, at the same time, the typical distance between two particles grows with time (as  $t^\mu$ ) and that the probability to find them at the same site tends to a finite constant? The physical picture is that of figure 4, where the probability  $P_n(t)$  is shown for a single sample and a fixed  $t$ . One sees that this probability distribution is made up of several sharp peaks that gather a finite fraction of the particles. However, the position of these peaks is scattered on a region of space of width  $t^\mu$ . As time progresses, the position and





**Figure 4.** Distribution of probability after a time  $t = 7 \times 10^{10} \overline{W}^{-1}$  for a particular sample of disorder in our one-dimensional directed random walk model with  $\mu = 0.4$ . The simulation was carried out with 1000 particles in a lattice of 20000 sites.

relative weights of these peaks of course change, but at any given (large) time only a finite number of peaks, corresponding to very large trapping times, contain most of the particles. This is clearly related to the fact that, for a given particle, most of its life was spent in the deepest ‘trap’ encountered up to time  $t$  [13, 14]. This behaviour is typical of the Lévy sums: when  $\mu < 1$  the sum of individual trapping times is dominated by the largest one. There is also a strong connection with the physics of the random energy model in its low-temperature (glassy) phase. The distribution of Boltzmann weights (which are also the residence times within each state) is again a power law with an exponent  $\mu < 1$ , whence only a finite (but random) number of states contribute to the full partition function [15, 16], and the probability that two independent copies of the same system occupy the same state is finite (and equal, on average, to  $y_2 = 1 - \mu$ ). As discussed in detail in [16], this is in turn related to ‘replica symmetry breaking’. All the  $y_q$ ’s can be computed and one finds, in particular,  $y_3 = y_2(y_2 + 1)/2$ . As explained above, we find analytically that  $y_2 \neq 1 - \mu$  in the dynamical model, which means that the system can never be considered in equilibrium, although the dynamics becomes slower with time [17]. At the same time, however, the equilibrium relation  $y_3 = y_2(y_2 + 1)/2$  appears to be fulfilled (see figure 3), suggesting that some kind of pseudo-equilibrium can be defined, for which equilibrium methods such as the replica method could be applied. It would be interesting to extend the method of the previous paragraph to calculate all the  $y_q$ ’s exactly, and to check whether they agree with the replica prediction.

Finally, it is interesting to note that the above biased model exhibits ageing effects when  $\mu < 1$  [18, 13, 19, 20]. In this context, a classification of different ageing models was proposed in [21], in terms of the asymptotic ‘clone overlap’ function. The idea is to look at the evolution of two identical systems (replicas), driven by the same thermal noise until  $t = t_w$ , and by independent thermal noise for later times. The two replicas can either separate with time (type I ageing), or remain close even after infinite time (type II ageing). One sees from the above example that, depending on the way in which one measures the ‘closeness’ of the two particles, one concludes differently. This situation is reminiscent of the quantum localization model introduced in [22], where states are (in certain regions of parameter space) both extended and localized, depending on the property which is studied. The physical nature of these quantum mixed states is actually very similar to the one discussed above.

## Acknowledgments

We thank A Baldassarri and M Mézard for useful discussions. AC thanks Professors José Casas-Vázquez and David Jou for their support and interest in this work. AC wishes to express his gratitude to the SPEC for their warm hospitality at CEA-Saclay and to the Direcció General de Recerca of the Generalitat de Catalunya for financial support within the programme FPI. Further funding has been obtained from the DGICyT of the Spanish Ministry of Education under grant no PB94-0718.

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